Math 335 Sample Problems

One notebook-sized page of notes (both sides may be used) will be allowed on the final exam. No electronic devices allowed. The final will be comprehensive.

1. Prove that
$$\int_0^1 x^{-x}dx = \sum_{n=1}^{\infty} n^{-n}.$$
Hint: Use the series expansion for e^x. Justify integrating term by term and compute, using the definition of the gamma function.

2. Prove that $f(r) = 1 - r + 2r^2 - 3r^3 + \cdots + (-1)^n nr^n + \ldots$ converges for $|r| < 1$. Does $\lim_{r \to 1^-} f(r)$ exist? If not, prove that; if it does exist, compute it.

3. Prove that
$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^{2/3}}$$
is not a Fourier series.

4. Suppose $\int_a^\infty f(x)dx$ converges absolutely. Prove that
$$\lim_{s \to \pm\infty} \int_a^\infty f(x)e^{isx}dx = 0.$$

5. Suppose f is continuous and piecewise smooth. Prove that
$$\sum_{n \neq 0} |\hat{f}((n)| \leq \left(2\sum_1^\infty \frac{1}{n^2}\right)^{1/2} \frac{1}{\sqrt{2\pi}} \left(\int_{-\pi}^{\pi} |f'|^2\right)^{1/2} = \sqrt{\frac{\pi}{6}} \left(\int_{-\pi}^{\pi} |f'|^2\right)^{1/2}$$

6. Prove that
$$\int_0^1 (1 - t^4)^{-1/2}dt = \frac{\Gamma(\frac{5}{4})\sqrt{\pi}}{\Gamma(\frac{3}{4})}.$$

7. Let f be a 2π-periodic function and let a be a fixed real number and let a new function g be defined by $g(x) = f(x - a)$. What is the relation between the Fourier coefficients $\widehat{f}(n)$ and $\widehat{g}(n)$?

8. Find the Fourier series of the following function of x.
$$\frac{1 - r^2}{1 - 2r\cos x + r^2}$$
where r is fixed with $0 \leq r < 1$. (You don't need to integrate.)

9. Let f be a 2π-periodic, piecewise smooth function. Let $\widehat{f}(n)$ be the complex Fourier coefficients of f. Show that there is a constant M (which will depend on f) such that $|\widehat{f}(n)| < M/|n|$ for all $n \neq 0$. Do **not** assume f is continuous.

10. Suppose f is Riemann integrable, and f_k is a sequence of Riemann integrable functions on $[0, 2\pi]$ such that $\lim_{k\to\infty} \int_0^{2\pi} |f_k - f| = 0$. Prove that the Fourier coefficients satisfy $\lim_{k\to\infty} \widehat{f_k}(n) = \widehat{f}(n)$ for each n.

11. Suppose f and g are 2π-periodic and Riemann integrable on compact subsets of $\mathbf{R}$. Suppose also that $f(x) = g(x)$ in a neighborhood of a point x_0. Suppose that the Fourier series for one of the functions converges at x_0. Prove that the other series converges at x_0 and

$$\sum_{-\infty}^{\infty} \widehat{f}(n)e^{inx_0} = \sum_{-\infty}^{\infty} \widehat{g}(n)e^{inx_0}.$$

Hint: Look at $f - g$.

12. Prove that

$$\lim_{n\to\infty} \int_0^{\pi} \frac{\sin(nx)}{x} dx \;=\; \frac{\pi}{2}.$$

13. Define a function $\log_p(x)$ inductively by the formulas $\log_0(x) = x, \log_{p+1}(x) = \log(\log_p(x))$. Prove by induction that the series

$$\sum_{n=m}^{\infty} \frac{1}{\log_0(n)\log_1(n)\log_2(n)\ldots\log_p(n)}$$

(where m is large enough for the denominators to be defined as real numbers) diverges for every p.

14. Suppose that $a_n > 0$, that a_n is decreasing, and that $\sum_1^{\infty} a_n$ converges. Is it true that $\lim_{n\to\infty} na_n = 0$? If true prove it, if false give a counterexample.

15. Suppose that f is 2π-periodic, continuous, and piecewise linear (that means that there is a finite set (in $[-\pi, \pi]$)of intervals in each of which f is defined by a linear function). Prove that

$$|\widehat{f}(n)| \leq \frac{c}{n^2},$$

for some constant c.

16. Show that the series $\sum_1^{\infty} \frac{\sin nx}{\sqrt{n}}$ converges for all x and uniformly on any interval of the form $[\delta, 2\pi - \delta]$, where $\delta > 0$ is small. Show that the series is not the Fourier series of a Riemann integrable function.

17. Find the solution of $u_t = 3u_{xx},\ u(0, t) = u(\pi, t) = 0,\ u(x, 0) = \cos x \sin 5x$. (This is easier than it looks.)

18. (a) Let $\sum_0^\infty a_n x^n$ be a series with radius of convergence R. Substitute $re^{i\theta}$ for x and get a new series involving $e^{in\theta}$. If $0 < r < R$ prove that this is a Fourier series (the variable is θ).

    (b) Prove that $\sum_0^\infty r^{2n}|a_n|^2$ converges for $0 \le r < R$.

19. Compute

$$\lim_{n\to\infty} \int_a^b \sin^2(nx)dx.$$

20. Let f and g be continuous 2π-periodic functions. Define the *convolution* of f and g to be the function. $f * g(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-t)g(t)dt$.

    (a) Prove that $f * g$ is 2π-periodic.

    (b) Prove that $\widehat{f * g}(n) = \widehat{f}(n)\widehat{g}(n)$, so the Fourier series of $f * g$ is $\sum_{-\infty}^{\infty} c_n d_n e^{inx}$, where $c_n = \widehat{f}(n),\ d_n = \widehat{g}(n)$.

21. (a) Find the cosine series of f where $f(x) = 0, 0 < x < \pi/2;\ f(x) = 1, \pi/2 < x < \pi$.

    (b) Prove that the series converges for all x.

    (c) For which x does the series converge absolutely?

22. Suppose $a_n > 0$ and $\sum_1^\infty a_n$ converges. Let $t_n = \sum_{k\ge n} a_k$.

    (a) Prove that $\sum \frac{a_n}{t_n}$ diverges.

    (b) Prove that $\sum \frac{a_n}{\sqrt{t_n}}$ converges.

23. Suppose $a_n > 0, b_n > 0$ suppose $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$ converge for all x. Suppose also that $\lim_{n\to\infty} \frac{a_n}{b_n} = c$. Prove that

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = c.$$

24. (a) Let $r = \sqrt{x^2+y^2}$. Prove that $\dfrac{y}{r^2}$ is harmonic when $y > 0$.

    (b) Suppose $\phi(t)$ is continuous on $[a, b]$. Let

$$u(x, y) = \int_a^b \frac{\phi(t) y dt}{(x-t)^2 + y^2}.$$

    Prove that u is harmonic when $y > 0$.

25. Suppose $f(x)$ is 2π periodic and satisfies $|f(x) - f(y)| \leq M|x - y|$ for all x, y. Let

$$u(r, \theta) = \int_0^{2\pi} f(\theta + \phi) P_r(\phi) d\phi,$$

for $0 < r < 1$, where $P_r(\phi)$ is the Poisson kernel. Prove that $\frac{\partial u}{\partial \theta}$ exists and $|\frac{\partial u}{\partial \theta}| \leq M$.

26. let f be 2π-periodic, continuous, and piecewise smooth. Let m be any positive integer and define the function f_m by the formula $f_m(x) = f(mx)$. Prove that $\widehat{f_m}(n) = \widehat{f}\left(\frac{n}{m}\right)$ if m divides n and is 0 otherwise.

27. There may be problems from the text, statements of theorems from the text, problems from previous review sets, or examples from class on the exam.